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Brownian Excursions on Combs

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In this paper we use techniques from Ito excursion theory to analyze Brownian motion on generalized combs. Ito excursion theory is a little-known area of probability theory and we therefore present a brief introduction for the uninitiated. A general method for analyzing transport along the backbone of the comb is demonstrated and the specific case of a comb whose teeth are scaling branching trees is examined. We then present a recursive method for evaluating the distribution of the first passage times on hierarchical combs.

KEY WORDS: Combs; trees; Brownian motion; Ito excursion theory; anomalous diffusion.

1. INTRODUCTION

Recently there has been considerable interest in diffusion in random environments, especially those which give rise to anomalous diffusion. One line of study has been the analysis of random walks on fractals,[1] where of course the anomalous diffusion is due to geometric effects rather than to the random environment. In order to gain insight into the problem of diffusion on fractals, random walks on generalized combs[2] have been studied where the transport along the backbone of the comb is the object of interest. The long-time behavior of random walks on both deterministic and random combs has been extensively studied using techniques from discrete random walks, mean field analysis, and scaling arguments.[3,4] More recently the first passage times of random walks on hierarchical combs have been studied using similar techniques and renormalization-semigroup analysis.[5]

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In this paper we examine Brownian motion on a periodic, deterministic generalized comb. First we present a general technique for analyzing transport along the backbone and then we look at the case where the teeth are scaling branching trees, i.e., where each branch gives rise to n offspring, ω times the length of the parent branch. We show that for $n^{-1} \leqslant \omega < n$ a diffusing particle can become lost in the tooth and hence the mean squared distance the particle moves along the backbone is asymptotically constant. In the case $\omega > n$ we find that, for large time t, the mean squared distance traveled along the backbone is log-asymptotic to $t^{(1-\alpha)/2}$, where $\alpha \equiv \log n/\log \omega$ and in the case $\omega < n^{-1}$ we find that the mean squared distance traveled goes as t. We then examine the distributions of first passage times on hierarchical combs, showing that the problem of calculating such a distribution reduces to the solution of a pair of nonlinear coupled difference equations. This paper contains a brief introduction to Ito excursion theory, which provides us with the main technique used in the analysis of the comb problem. Rogers[6] and Rogers and Williams[7] give an excellent introduction to excursion theory, also giving a general method for calculating the excursion rates used in this paper.

It is of practical importance to understand the mechanisms responsible for anomalous diffusion in random systems, for example, dispersion of a pollutant in the environment. The comb model exhibits some of the characteristics of dispersion through fractured rock systems,[8] where the backbone corresponds to a main crack along which we are interested in the dispersion. The comb model also has been examined to gain insight into flow through porous media.[9]

2. ITO EXCURSION THEORY

This section is a brief introduction to Ito excursion theory for Brownian motion for those new to the subject. Excursion ideas apply to any Markov process with a recurrent state; for a detailed development of the theory see Rogers and Williams.[7]

Consider a standard Brownian motion $(B_t)_{t \geqslant 0}$ on $\mathbb{R}$ starting at 0; then by the continuity of B we can write $\{t: B_t \neq 0\}$ as the disjoint countable union of maximal open intervals, i.e., $\bigcup_i (a_i, b_i)$. During each of these intervals B makes an *excursion* away from 0. An excursion can be considered as a point in the excursion space U, where

$$U \equiv \{f \mid f \text{ is a continuous function from } \mathbb{R}^+ \text{ to } \mathbb{R}$$

$$\text{and } f^{-1}(\mathbb{R} \backslash \{0\} \text{ is an open interval}\}$$

The length of the interval $f^{-1}(\mathbb{R}\backslash\{0\})$ is the duration of the excursion. So an excursion for Brownian motion has the form

$$\{B_{\min(t+a_i, b_i)}: t \geqslant 0\}$$

The order of the excursions is parametrized by local time.

Theorem (Trotter). There exists a jointly continuous process $\{L(t, x): t \geqslant 0, x \in \mathbb{R}\}$ such that for all bounded, measurable f and all $t \geqslant 0$

$$\int_0^t f(B_s)\, ds = \int_{-\infty}^{\infty} f(x)\, L(t, x)\, dx$$

The function $L(t, \cdot)$ is the occupation time density, called the local time. A heuristic definition of L, useful for calculations, is

$$L(t, x) \equiv \int_0^t \delta(B_s - x)\, ds \tag{2.1}$$

where δ is the one-dimensional Dirac delta function with unit mass at 0. Hence if $b < a'$, then the excursions occurring in the intervals (a, b) and (a', b') occur at local times at 0, l, and l' respectively, where $l < l'$. Using this, we can split a Brownian sample path apart into its excursions from 0 and represent a Brownian path as a point process Ξ in $\mathbb{R}^+ \times U$, where $(l, \xi) \in \Xi$ if and only if the Brownian motion makes excursion ξ at local time l. This procedure can be reversed, as, if we know Ξ, we can join the excursions in the correct order to recover the original sample path. The fundamental result of excursion theory is as follows:

Theorem (Ito). The excursion point-process is Poisson with excursion measure Lebesgue $\times n$, where the σ-finite measure n on U is called the excursion measure.

Thus we deduce:

(i) If $A \subset U$, $0 \leqslant n(A) < \infty$, then

$$\mathbb{P}(\Xi \text{ has no point in } (0, l) \times A) = e^{-ln(A)}$$

(ii) If $A_1, ..., A_k \subset U$ are disjoint, $0 \leqslant n(A_i) < \infty$ for all i, $A = \bigcup_1^k A_i$, and $n(A) > 0$, then

$$\mathbb{P}(\Xi_l \in A_i) = n(A_i)/n(A)$$

where $l \equiv \inf\{u: \Xi_u \in A\}$.

We are now faced with the problem of calculating the excursion measure n of some set A. Ito's theorem means, however, that the local time at which a first excursion in A occurs is exponentially distributed with rate $n(A)$. Hence, for a Brownian motion started at 0, we have

$$n(A) = (\mathbb{E}^0(L_A))^{-1}$$

where L_A is the local time at 0 at which the first excursion in A occurs. This can be found by calculating the expected local time spent at 0 by a Brownian motion which is killed when it has an excursion in A. There are many elegant methods for calculating excursion rates, but the method we shall use involves only standard mathematical methods.

Examples. Rates of Hitting Levels. Let A be the set of excursions hitting the point $-a$ or a. To find $n(A)$, we calculate the expected local time at 0 of a Brownian motion killed at $\pm a$; this expected local time is therefore $n(A)^{-1}$.

For a Brownian motion started at $x \in [-a, a]$ and killed at $\pm a$, define $\phi(x)$ to be the expected local time at 0 before the process is killed. It is easy to show, using the definition (2.1), that ϕ satisfies

$$\frac{1}{2}\frac{d^2\phi}{dx^2} + \delta(x) = 0 \tag{2.2}$$

with the (killing) boundary conditions $\phi(-a) = \phi(a) = 0$. Solving (2.2) yields $\phi(0) = a$, giving $n(A) = a^{-1}$. Excursions hitting $-a$ and a, by symmetry, occur at the same rate and are also disjoint, hence

$$n(\text{hit } -a) = n(\text{hit } a) = 1/2a \tag{2.3}$$

Marked Excursions. In the following examples we mark the sample paths at an exponentially distributed real time of rate $\frac{1}{2}\alpha^2$, independent of the path. These marked excursions, strictly, belong to a richer excursion space,[7] however, for our purposes we need not worry about the precise details here.

(i) Let A be the set of excursions which are marked or hit $\pm a$. Therefore $n(A)^{-1}$ is equal to the expected local time spent at 0 by a Brownian motion which is killed at $\pm a$ and at real time rate $\frac{1}{2}\alpha^2$. In this case the equation for the expected local time at 0 for a Brownian motion started at $x \in [-a, a]$ and killed at rate $\frac{1}{2}\alpha^2$ is

$$\frac{1}{2}\frac{d^2\phi}{dx^2} - \frac{1}{2}\alpha^2\phi + \delta(x) = 0 \tag{2.4}$$

with the (killing) boundary conditions $\phi(-a)=\phi(a)=0$. Solving (2.4) yields $\phi(0)=\alpha^{-1}\tanh(\alpha a)$. Hence $n(A)=\alpha\coth(\alpha a)$ and using the symmetry between upward and downward excursions, we obtain

$$n(\text{hit } -a \text{ or marked}) = n(\text{hit } a \text{ or marked}) = \tfrac{1}{2}\alpha\coth(\alpha a) \tag{2.5}$$

In the limit $a \to \infty$, (2.5) becomes

$$n(\text{marked in } [-\infty, 0]) = n(\text{marked in } [0, \infty]) = \tfrac{1}{2}\alpha \tag{2.6}$$

[The reader might like to check (2.6) by direct calculation!]

(ii) If A is the set of excursions which hit a before they are marked, then we can use the Poisson nature of the excursions to find

$$\mathbb{P}^0(\text{hit } a \text{ before marked}) = \frac{n(A)}{n(B)+n(C)}$$

where B is the set of upward excursions hitting a or marked and C is the set of marked downward excursions. Since

$$\mathbb{P}^0(\text{hit } a \text{ before marked}) = e^{-\alpha a}$$

from (2.5) and (2.6) we find

$$e^{-\alpha a} = \frac{n(A)}{\frac{1}{2}\alpha\coth(\alpha a) + \frac{1}{2}\alpha}$$

giving

$$n(A) = \tfrac{1}{2}\alpha\,\mathrm{cosech}(\alpha a) \tag{2.7}$$

(iii) If the process is a Brownian motion reflected at $\pm a$, then the local-time rate of marked excursions is given by a measure m on excursions of reflected Brownian motion. If $\phi(x)$ is the expected local time at 0 for this process started at $x \in [-a, a]$, then $\phi(x)$ satisfies (2.4) subject to the (reflecting) boundary conditions $\phi'(\pm a)=0$. This gives $\phi(0)=\alpha^{-1}\coth(\alpha a)$. Hence, using the symmetry between upward and downward excursions, we find

$$m(\text{marked in } [-a, 0]) = m(\text{marked in } [0, a]) = \tfrac{1}{2}\alpha\tanh(\alpha a) \tag{2.8}$$

3. BASIC FORMALISM

Consider a comb consisting of a copy of $\mathbb{R}$, the backbone, with teeth attached at a set of roots $(2na)_{n\in\mathbb{Z}}$, where $a>0$ (see Fig. 1). We shall refer

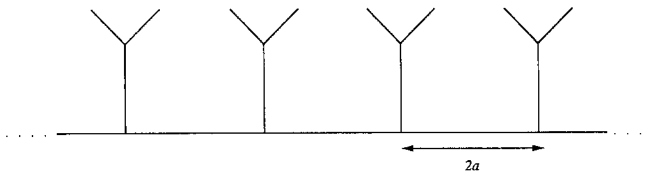

Fig. 1. An example of a generalized comb with Y-shaped teeth.

to any point where the comb branches as a node (so a root is also a node). Away from a node the particle diffuses as a standard Brownian motion and at a node the rates of going in any direction are equal; we denote this process on the comb by $B \equiv (B_t)_{t \geqslant 0}$. We define the process $W \equiv (W_t)_{t \geqslant 0}$ on the backbone to be B_t when B_t is on the backbone and to be the root of the tooth when B_t is in a tooth. Hence W is a random time change of a Brownian motion on the backbone (and is in fact a martingale). Consequently, if $T(t)$ is the time spent by B on the backbone until real time t, then $\exp[i\theta W_t + \frac{1}{2}\theta^2 T(t)]$ is a martingale. Therefore, if we mark the process B at an exponentially distributed real time M, independent of the path and of rate $\mu \equiv \frac{1}{2}\sigma^2$, then

$$\mathbb{E}^o(\exp[i\theta W_M + \tfrac{1}{2}\theta^2 T(M)]) = 1 \tag{3.1}$$

where the superscript on the expectation denotes the starting point. The characteristic function for W_M cannot be obtained directly from (3.1), as W_M and $T(M)$ are dependent. However, considering the coefficient of θ^2 in (3.1) gives

$$\mathbb{E}^o(W_M^2) = \mathbb{E}^o(T(M)) \tag{3.2}$$

We calculate

$$\psi(\theta, \sigma) \equiv \mathbb{E}^o(e^{-\theta^2 T(M)/2})$$

by considering a diffusion $\tilde{B} \equiv (\tilde{B}_t)_{t \geqslant 0}$ on a graph G which is the interval $K \equiv [-a, a]$ with a tooth attached at the origin O. The diffusion $\tilde{B}$ is a standard Brownian motion except at the nodes, where its rates of going in any direction are equal, and at the points $\pm a$, which are reflecting. The periodicity of the original comb means that $T(t)$ is identical in distribution to the total real time spent on the lower segment K of G by $\tilde{B}$ until real time t. We mark $\tilde{B}$ at an exponentially distributed real time $\tilde{M}$, of rate μ, independent of the path, and at an exponentially distributed real time N spent on K, independent of M, of rate $\nu \equiv \frac{1}{2}\theta^2$; therefore

$$\psi(\theta, \sigma) = \mathbb{P}^o(\mu \text{ mark before } \nu \text{ mark})$$

By the definitions of $\tilde{M}$ and N, μ marking occurs everywhere on G and ν marking occurs on K only. Hence the real-time rate of all marking on K is

$\kappa = \mu + \nu$ and in the tooth is μ. If n is the excursion measure on Brownian motion on $[0, a]$ reflected at 0 and a, and m is the excursion measure on the process B in the tooth with reflection at O, then the local-time rate of marked excursions of $\tilde{B}$ on G is given by

$$\tfrac{1}{3} m^o(\mu \text{ marked}) + \tfrac{2}{3} n^o(\kappa \text{ marked})$$

where the superscript o indicates the starting point of the excursion. (As we are concerned with the ratios of local-time rates from the nodes and not the normalization of local time at the nodes, we could dispense with the factors of 1/3 providing we did this consistently.) The local-time rate of all excursions μ marked before ν marked is

$$\tfrac{1}{3} m^o(\mu \text{ marked}) + \tfrac{2}{3} n^o(\kappa \text{ marked})\, p$$

where

$$p \equiv \mathbb{P}^o(\mu \text{ mark before } \nu \text{ mark} \mid \kappa \text{ marked})$$

However, on K the two marking processes are independent Poisson processes of rates μ and ν; therefore

$$p = \frac{\mu}{\mu + \nu}$$

Using the Poisson nature of excursions of $\tilde{B}$ on G, we deduce

$$\psi(\theta, \sigma) = \frac{\tfrac{1}{2} m^o(\mu \text{ marked}) + n^o(\kappa \text{ marked})\, p}{\tfrac{1}{2} m^o(\mu \text{ marked}) + n^o(\kappa \text{ marked})}$$

From (2.8)

$$n^o(\kappa \text{ marked}) = (\sigma^2 + \theta^2)^{1/2} \tanh[(\sigma^2 + \theta^2)^{1/2}\, a]$$

Hence

$$\psi(\theta, \sigma) = \frac{\tfrac{1}{2} k(\sigma) + [\sigma^2/(\sigma^2 + \theta^2)^{1/2}] \tanh[(o^2 + \theta^2)^{1/2}\, a]}{\tfrac{1}{2} k(\sigma) + (\sigma^2 + \theta^2)^{1/2} \tanh[(\sigma^2 + \theta^2)^{1/2}\, a]} \tag{3.3}$$

where $k(\sigma) \equiv m^o(\mu \text{ marked})$.

We can also calculate the probability that B is in a tooth at time M by noticing that it is equal to the probability that $\tilde{B}$ gets its first μ mark in the tooth, i.e.,

$$\mathbb{P}^0(\mu \text{ marked in a tooth before } \mu \text{ marked on the backbone})$$
$$= \frac{\tfrac{1}{2} k(\sigma)}{\tfrac{1}{2} k(\sigma) + \sigma \tanh(\sigma a)} \tag{3.4}$$

The Characteristic Function for W_M

If the process B is marked at an exponentially distributed random time M, independent of the path and of rate $\mu \equiv \frac{1}{2}\sigma^2$, we may write

$$W_M = 2aJ_M + Z_M$$

where $J_M \in \mathbb{Z}$ and $2aJ_M$ is the last root visited by the process B before it is marked; Z_M is the additional distance that W travels from $2aJ_M$ before it is marked and, by definition, therefore, $Z_M \in (-2a, 2a)$. The memoryless property of the exponential time M and the periodicity of the comb ensure that Z_M is independent of J_M; therefore

$$\mathbb{E}^o(e^{i\theta W_M}) = \mathbb{E}^o(e^{2ia\theta J_M})\, \mathbb{E}^o(e^{i\theta Z_M}) \tag{3.5}$$

The local-time rate of excursions from a given root which hit a neighboring root before being marked is $\frac{2}{3}\sigma \operatorname{cosech}(2\sigma a)$ from (2.7). Excursions which hit a neighboring root or are marked have local-time rate $\frac{2}{3}\sigma \times \coth(2\sigma a) + \frac{1}{3}k(\sigma)$, where we have used (2.5) and recall that $k(\sigma)$ is defined to be the local time rate of marked excursions from the root into the tooth for the process B if it is reflected at O. If $p(\sigma)$ is defined to be the probability that W moves from one root to a neighboring root without a mark, then the Poisson nature of the excursions of B leads to the result

$$p(\sigma) = \frac{\sigma \operatorname{cosech}(2\sigma a)}{\sigma \coth(2\sigma a) + \frac{1}{2}k(\sigma)}$$

The probability generating function for J_M is given by

$$\begin{aligned} G(s, \sigma) &= \sum_{i=0}^{\infty} [1 - p(\sigma)]\, p(\sigma)^i \frac{1}{2^i} (s + s^{-1})^i \\ &= \frac{1 - p(\sigma)}{1 - \frac{1}{2} p(\sigma)(s + s^{-1})} \end{aligned}$$

Hence

$$\mathbb{E}^o(e^{2ia\theta J_M}) = \left(1 + \frac{2 \sin^2(a\theta)\, \sigma \operatorname{cosech}(2\sigma a)}{\frac{1}{2}k(\sigma) + \sigma \tanh(\sigma a)}\right)^{-1} \tag{3.6}$$

If the comb had no teeth, then W would simply be a standard Brownian motion $\tilde{W}$ on $\mathbb{R}$ and

$$\begin{aligned} \mathbb{E}^o(e^{2ia\theta J_M}) &= \mathbb{E}^o(e^{-\theta^2 M/2}) \\ &= \frac{\sigma^2}{\theta^2 + \sigma^2} \end{aligned} \tag{3.7}$$

In this instance Z is simply a Brownian motion $\tilde{Z}$ started at 0 and conditioned to be marked before hitting $\pm 2a$. Using (3.5) and (3.6) with $k(\sigma)=0$, we conclude

$$\mathbb{E}^{o}(\exp(i\theta \tilde{Z}_{M}))=\frac{\sigma^{2}}{\theta^{2}+\sigma^{2}}\left(\frac{\cosh(2a\sigma)-\cos(2a\theta)}{\cosh(2a\sigma)-1}\right)$$

For a comb, however,

$$\mathbb{E}^{o}(e^{i\theta Z_{M}})=\mathbb{E}^{o}(e^{i\theta Z_{M}};\text{ marked in a tooth})+\mathbb{E}^{o}(e^{i\theta Z_{M}};\text{ marked on backbone})$$

Using (3.4),

$$\mathbb{E}^{o}(e^{i\theta Z_{M}};\text{ marked in a tooth})=\mathbb{P}^{o}(\text{marked in a tooth})=\frac{\frac{1}{2}k(\sigma)}{\sigma\tanh(\sigma a)+\frac{1}{2}k(\sigma)}$$

and

$$\begin{aligned}&\mathbb{E}^{o}(e^{i\theta Z_{M}};\text{ marked on backbone})\\&\quad=\mathbb{P}^{o}(\text{marked on backbone})\,\mathbb{E}^{o}(e^{i\theta Z_{M}}\mid\text{marked on backbone})\\&\quad=\frac{\sigma\tanh(\sigma a)}{\sigma\tanh(\sigma a)+\frac{1}{2}k(\sigma)}\,\mathbb{E}^{o}(e^{i\theta Z_{M}}\mid\text{marked on backbone})\end{aligned}$$

If Z' is the process Z conditioned on being marked on the backbone, then Z'_{M} is identically distributed to $\tilde{Z}_{M}$ and therefore

$$\begin{aligned}\mathbb{E}^{o}(e^{i\theta Z_{M}})=&\frac{\frac{1}{2}k(\sigma)}{\sigma\tanh(\sigma a)+\frac{1}{2}k(\sigma)}\\&+\frac{\sigma\tanh(\sigma a)}{\sigma\tanh(\sigma a)+\frac{1}{2}k(\sigma)}\left(\frac{\sigma^{2}}{\theta^{2}+\sigma^{2}}\right)\left(\frac{\cosh(2a\sigma)-\cos(2a\theta)}{\cosh(2a\sigma)-1}\right)\end{aligned}\tag{3.8}$$

[This result can also be obtained by calculating the probability distribution function for Z_{M} on $(-2a, 2a)$ via a rather involved excursion argument; the characteristic function is then obtained by taking the Fourier transform of the corresponding probability density function.] Combining (3.8) and (3.6), we obtain

$$\mathbb{E}^{o}(e^{i\theta W_{M}})=\frac{\frac{1}{2}k(\sigma)+\left(\dfrac{\sigma^{3}\tanh(\sigma a)[\cosh(2a\sigma)-\cos(2a\theta)]}{\{(\theta^{2}+\sigma^{2})[\cosh(2a\sigma)-1]\}}\right)}{\sigma\tanh(\sigma a)+\frac{1}{2}k(\sigma)+2\sin^{2}(a\theta)\operatorname{cosech}(2\sigma a)}\tag{3.9}$$

For any periodic comb, once we have calculated $k(\sigma)$ corresponding to the tooth type, the Fourier-Laplace transform for the probability density function for the distance the particle has moved along the backbone follows from (3.9).

Example. Havlin and Weiss[4] calculated the aymptotic behavior of $\mathbb{E}^o(W_t^2)$ at large time for a comb whose teeth were lines of infinite length; this result can be immediately recovered as follows. From (3.2) and (3.3) we find

$$\mathbb{E}^o(W_M^2) = \frac{2}{\sigma}\left(\frac{\tanh(\sigma a)}{\frac{1}{2}k(\sigma) + \sigma\tanh(\sigma a)}\right) \tag{3.10}$$

In this case $k(\sigma) = \sigma$ from (2.6). Note

$$\mathbb{E}^o(W_M^2) = \int_0^\infty \mu e^{-\mu t}\mathbb{E}^o(W_t^2)\,dt \tag{3.11}$$

and as we are interested in the long-time behavior, we expand about $\mu = 0$ and invert the Laplace transform to obtain

$$\mathbb{E}^o(W_t^2) = 4a\left(\frac{2t}{\pi}\right)^{1/2} - 8a^2 + \frac{44}{3}\left(\frac{2}{\pi t}\right)^{1/2} a^3 + O(t^{-1}) \qquad \text{as} \quad t \to \infty$$

4. COMBS WITH BRANCHING TEETH

Let each tooth be a branching tree, where each branch of the tree gives rise to n offspring, with $n > 1$, of length ω times the parent branch (see Fig. 2). The length of the original branch attached to the root is c and in the case $\omega < 1$ we take the tips of the tree to be reflecting. We shall see that for $n^{-1} \leqslant \omega < n$ there is a nonzero local-time rate at the roots for the

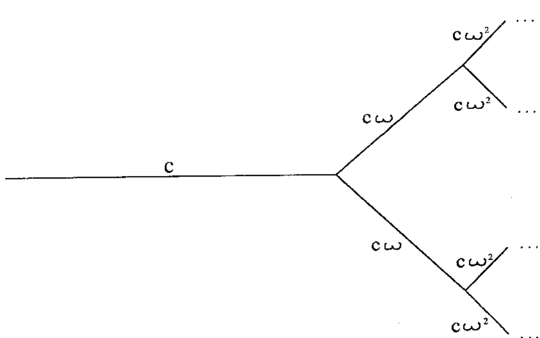

Fig. 2. Scaling branching tree with $n = 2$.

particle to leave K for ever. The *radial* part of the diffusion on the tree (i.e., the distance from the root) reflected at O is a diffusion $(X_t)_{t \geq 0}$ on $[0, x_\infty]$, where $x_\infty = c/(1-\omega)$ for $\omega < 1$ and $x_\infty = \infty$ for $\omega \geq 1$. The particle X receives kicks away from the root 0 at the points $x_j \equiv c(1-\omega^j)/(1-\omega)$, $j \in \mathbb{N}$, and the effect of these kicks is to make the rate of upward excursions from the x_j equal to n times the rate of the downward excursions. When considering the radial part alone, it is useful for technical reasons to take n to be a continuous parameter in $[1, \infty]$. The local-time rate of nonreturning excursions $\gamma(c, \omega, n)$ of X is, by the Markov property of the excursions, given by

$$\gamma(c, \omega, n) = m^0(\text{hit } x_1) \cdot \mathbb{P}^{x_1}(\text{not hit } 0)$$

as all nonreturning excursions hit x_1 almost surely. From (2.3)

$$m^0(\text{hit } x_1) = 1/c$$

The generator for the diffusion X is

$$G \equiv \frac{1}{2}\frac{d^2}{dx^2} + \left(\frac{n-1}{n+1}\right) \sum_{j=1}^{\infty} \delta(x - x_j) \frac{d}{dx}$$

Putting the diffusion X into natural scale,[7] i.e., writing $Y_t \equiv s(X_t)$, where $Gs(x) = 0$, we find

$$s'(x) = n^{-j} \qquad \text{for} \quad x \in (x_j, x_{j+1})$$

and therefore $s(0) = 0$, $s(x_1) = c$, and

$$\begin{aligned} s(x_\infty) &= \sum_0^\infty c \left(\frac{\omega}{n}\right)^j \\ &= c\left(1 - \frac{\omega}{n}\right)^{-1} \qquad \text{for} \quad \omega < n \\ &= \infty \qquad \text{for} \quad n \leq \omega \end{aligned}$$

However,

$$\mathbb{P}^{x_1}(\text{hit } x_\infty \text{ before } 0) = \frac{s(x_1) - s(0)}{s(x_\infty) - s(0)}$$

giving

$$\begin{aligned} \mathbb{P}^{x_1}(\text{hit } x_\infty \text{ before } 0) &= \left(1 - \frac{\omega}{n}\right) \qquad \text{for} \quad \omega < n \\ &= 0 \qquad \text{for} \quad n \leq \omega \end{aligned}$$

In the case $\omega \geqslant 1$ the sample paths from x_1 that never hit 0 are almost surely those that go to infinity before hitting 0; however, in the case $\omega < 1$ paths from x_1 hitting x_∞ before 0 do so in finite time with probability 1 and hence can only become lost if they are trapped at x_∞. If the endpoints of the tree are trapping, then $s(x_\infty)$ is an exit boundary point[7] for the diffusion Y, namely

$$\int_0^{s(x_\infty)} m(dy) = \infty$$

where $m(dy) \equiv (s'(x))^{-2}\, dy$ is the speed measure for Y. Therefore $s(x_\infty)$ is an exit boundary point when

$$\int_0^{x_\infty} \frac{dx}{s'(x)} = \sum_{j=0}^{\infty} c(n\omega)^j = \infty$$

namely when $n\omega \geqslant 1$. Hence, despite the fact that we are modeling a physical system that has "reflecting endpoints," our mathematical model makes them exit points in the strict probabilistic sense. Thus we conclude

$$\begin{aligned}
\gamma(c, \omega, n) &= 0 && \text{for} \quad \omega < n^{-1} \\
&= \frac{1 - \omega/n}{c} && \text{for} \quad n^{-1} \leqslant \omega < n \\
&= 0 && \text{for} \quad n \leqslant \omega
\end{aligned}$$

The Case $n^{-1} \leqslant \omega < n$. In this case there is a nonzero local-time rate at the roots at which the particle is lost from the backbone. At this random time $\mathscr{L}$ the diffusion along the backbone ceases and the process W is constant for all $t > \mathscr{L}$. Setting $\sigma = 0$ in (3.9) and noting $k(0, c, \omega, n) = \gamma(c, \omega, n)$ gives

$$\mathbb{E}^o(e^{i\theta W_{\mathscr{L}}}) = \frac{\frac{1}{2}a\gamma(c, \omega, n)}{\frac{1}{2}a\gamma(c, \omega, n) + \sin^2(a\theta)}$$

so

$$\mathbb{E}^o(W_{\mathscr{L}}^2) = \frac{4ca}{1 - \omega/n}$$

Functional Equation for k. For the radial diffusion X, with real $n \geqslant 1$, we determine a functional equation for k, the local-time rate of marked excursions of X from 0 into $[0, x_\infty]$. All μ marked excursions of

X from 0 are either marked before hitting x_1 or hit x_1 without a mark but are marked before returning to 0; therefore

$$\begin{aligned}&k(\sigma, c, \omega, n)\\&\quad = m^0(\mu \text{ mark before hit } x_1)\\&\qquad + m^0(\text{hit } x_1 \text{ before } \mu \text{ mark})[1 - \mathbb{P}^{x_1}(\text{hit } 0 \text{ before } \mu \text{ mark})]\end{aligned}$$

However,

$$\begin{aligned}&m^0(\mu \text{ mark before hit } x_1)\\&\quad = m^0(\mu \text{ mark or hit } x_1) - m^0(\text{hit } x_1 \text{ before } \mu \text{ mark})\end{aligned}$$

giving

$$\begin{aligned}&k(\sigma, c, \omega, n)\\&\quad = m^0(\mu \text{ mark or hit } x_1)\\&\qquad - m^0(\text{hit } x_1 \text{ before } \mu \text{ mark})\, \mathbb{P}^{x_1}(\text{hit } 0 \text{ before } \mu \text{ mark})\end{aligned}$$

From x_1 the excursion can either return to 0 without a mark or can be μ marked moving up from x_1 or returning to 0; therefore, using the Poisson nature of excursions from x_1, we find

$$\begin{aligned}&\mathbb{P}^{x_1}(\text{hit } 0 \text{ before } \mu \text{ mark})\\&\quad = \frac{m^{x_1}(\text{hit } 0 \text{ before } \mu \text{ mark})}{nk(\sigma, c\omega, \omega, n) + m^{x_1}(\mu \text{ mark in } [0, x_1] \text{ or hit } 0)}\end{aligned}$$

where m^{x_1} is the excursion measure on excursions of Brownian motion on $[0, x_1]$ started at x_1 and reflected at 0 and x_1. From (2.5) we find

$$m^{x_1}(\mu \text{ mark in } [0, x_1] \text{ or hit } 0) = m^0(\mu \text{ mark or hit } x_1) = \sigma \coth(\sigma c)$$

and from (2.7)

$$m^{x_1}(\text{hit } 0 \text{ before } \mu \text{ mark}) = m^0(\text{hit } x_1 \text{ before } \mu \text{ mark}) = \sigma \operatorname{cosech}(\sigma c)$$

Hence

$$k(\sigma, c, \omega, n) = \sigma \coth(\sigma c) - \frac{\sigma^2 \operatorname{cosech}^2(\sigma c)}{nk(\sigma, c\omega, \omega, n) + \sigma \coth(\sigma c)} \tag{4.1}$$

By dimensional argument we find

$$k(\sigma, c, \omega, n) = \sigma \lambda(\sigma c, \omega, n)$$

for some function λ, and (4.1) implies that λ satisfies the following functional equation:

$$\lambda(\sigma c, \omega, n) = \frac{n\lambda(\sigma c\omega, \omega, n) + \tanh(\sigma c)}{n\lambda(\sigma c\omega, \omega, n)\tanh(\sigma c) + 1} \tag{4.2}$$

As $c \to \infty$, the tooth becomes an infinite line, as in the previous example; hence

$$\lim_{c \to \infty} \lambda(\sigma c, \omega, n) = 1 \tag{4.3}$$

because $\lim_{c \to \infty} k(\sigma, c, \omega, n) = \sigma$. In the case $\omega = 1$, we can solve (4.2). Taking the positive root, we obtain

$$k(\sigma, c, 1, n) = \frac{\sigma(n-1)}{2n\tanh(\sigma c)}\left[1 + \left(1 + \frac{4n\tanh^2(\sigma c)}{(n-1)^2}\right)^{1/2}\right]$$

For small σ this gives

$$\mathbb{E}^o(W_M^2) \to \frac{4nca}{n-1} \qquad \text{as} \qquad \sigma \to 0$$

and therefore

$$\mathbb{E}^o(W_t^2) \to \frac{4nca}{n-1} \qquad \text{as} \qquad t \to \infty$$

The Case $\omega \geqslant n$. Given suitable assumptions about the solution of (4.2) near $\sigma = 0$, for fixed ω and n with $\omega \geqslant n$, we can show (see Appendix) that

$$\mathbb{E}^o(W_t^2) \sim ac\left(\frac{t}{c^2}\right)^{(1-\alpha)/2} g\left(\frac{t}{c^2}, \omega, n\right) \qquad \text{as} \quad t \to \infty \tag{4.4}$$

where (4.4) is (A.8) in dimensional form, $\alpha = \log n/\log \omega$, and $g(\cdot, \omega, n)$ is a slowly varying function[10] (at infinity).

When $\omega > n$ a slightly weaker form of (4.4), in terms more familiar to physicists, can be found by using Proposition 1.3.6 of Bingham *et al.*,[10] namely

$$(ac)^{-1}\,\mathbb{E}^o(W_t^2) \approx \left(\frac{t}{c^2}\right)^{(1-\alpha)/2}$$

where $\psi \approx \phi$ means $\log \phi(t) \sim \log \psi(t)$ as $t \to \infty$. Thus we have a system that exhibits a range of anomalous diffusive behavior.

The Case $\omega < n^{-1}$. In the Appendix we show that, in this parameter regime,

$$k(\sigma, c, \omega, n) \sim \frac{\sigma^2 c}{1 - n\omega} \qquad \text{as} \quad \sigma \to 0 \tag{4.5}$$

Substituting (4.5) into (3.10) gives

$$\mathbb{E}^o(W_M^2) \sim \frac{2a}{\mu[2a + c/(1 - n\omega)]} \qquad \text{as} \quad \mu \to 0$$

and therefore

$$\mathbb{E}^o(W_t^2) \sim \frac{2at}{2a + c/(1 - n\omega)} \qquad \text{as} \quad t \to \infty \tag{4.6}$$

This also follows easily from an ergodic argument: for large t the proportion of time spent on K tends (almost surely) to $|K|/|G|$, giving $\mathbb{E}^o(W_t^2) \sim |K|\, t/|G|$.

5. FIRST PASSAGE TIMES FOR HIERARCHICAL COMBS

In this section we consider the distributions for first passage times for Brownian motion on hierarchical combs; we show that the problem can be reduced to a pair of coupled nonlinear difference equations.

Following Kahng and Redner,[5] we take the roots to be a distance 1 apart and the ratio of the tooth lengths between successive iterations to be R. The jth-order comb C_j is formed by attaching the free ends of the backbones of two $(j-1)$th-order combs at 0 and then attaching a tooth of length R^{j-1} at 0; the zeroth-order comb is simply a unit line segment. The midpoint of C_j is therefore at 0 for all j and the free ends of its backbone are at $\pm 2^{j-1}$. We consider a process $(B_t)_{t \geqslant 0}$ on C_j, with reflecting endpoints, which is a standard Brownian motion except at the nodes, where its rates of going in any direction (on the comb) are equal. Let T_j be the first time for B to travel from one end of the backbone to the other, i.e., from $\mp 2^{j-1}$ to $\pm 2^{j-1}$. If we mark B at an exponentially distributed real time M, independent of its path and of rate $\mu \equiv \frac{1}{2}\sigma^2$, then

$$\mathbb{E}(e^{-\sigma^2 T_j/2}) = p_j(\sigma)$$

where

$$\begin{aligned} p_j(\sigma) &= \mathbb{P}^{-2^{j-1}}(T_j < M) \\ &= \mathbb{P}^{-2^{j-1}}(\text{hit } 2^{j-1} \text{ before marked}) \end{aligned}$$

If $u_j(\sigma)$ is the local-time rate of excursions from -2^{j-1} that hit 2^{j-1} without a mark and $v_j(\sigma)$ is the local-time rate of those that hit 2^{j-1} or are marked, then

$$p_j(\sigma)=\frac{u_j(\sigma)}{v_j(\sigma)}$$

We find a recursive method for calculating these rates as follows. By the Markov property of the excursions,

$$u_j(\sigma)=u_{j-1}(\sigma)\cdot\mathbb{P}^0(\text{hit } 2^{j-1} \text{ before marked or returning to } -2^{j-1})$$

since an excursion hitting 2^{j-1} without a mark must first hit 0 without a mark. However,

$$\mathbb{P}^0(\text{hit } 2^{j-1} \text{ before marked or returning to } -2^{j-1})$$
$$=\frac{u_{j-1}(\sigma)}{2v_{j-1}(\sigma)+\sigma\tanh(\sigma R^{j-1})}$$

giving

$$u_j(\sigma)=\frac{u_{j-1}^2(\sigma)}{2v_{j-1}(\sigma)+\sigma\tanh(\sigma R^{j-1})} \tag{5.1}$$

Furthermore,

$$v_j(\sigma)=v_{j-1}(\sigma)-u_{j-1}(\sigma)+u_{j-1}(\sigma)\,\mathbb{P}^0(\text{marked or hit } 2^{j-1} \text{ before hit } -2^{j-1})$$

However,

$$\mathbb{P}^0(\text{marked or hit } 2^{j-1} \text{ before hit } -2^{j-1})=1-\frac{u_{j-1}(\sigma)}{2v_{j-1}(\sigma)+\sigma\tanh(\sigma R^{j-1})}$$

giving

$$v_j(\sigma)=v_{j-1}(\sigma)-\frac{u_{j-1}^2(\sigma)}{2v_{j-1}(\sigma)+\sigma\tanh(\sigma R^{j-1})} \tag{5.2}$$

As the zeroth-order comb is just a unit line segment, (2.5) and (2.7) give the initial conditions

$$u_0(\sigma)=\sigma\operatorname{cosech}(\sigma),\qquad v_0(\sigma)=\sigma\coth(\sigma) \tag{5.3}$$

Therefore Eqs. (5.1) and (5.2) with initial conditions (5.3) provide a method for calculating the distribution of the T_j for all $j\geqslant 0$.

APPENDIX

Here we shall without loss of generality take c, the length of the first branch of the tree, to be unity. Near $\sigma = 0$ we expect $\lambda(\sigma, \omega, n)$ to be close to a power law in σ; therefore to proceed we assume that

$$\lambda(\sigma, \omega, n) = \sigma^{-\alpha} f(\sigma^{-1}, \omega, n) \tag{A.1}$$

where $f(\cdot, \omega)$ is measurable and slowly varying (at infinity) for fixed ω and n, i.e.,

$$\lim_{\sigma \to 0} \frac{f(\tau\sigma^{-1}, \omega, n)}{f(\sigma^{-1}, \omega, n)} = 1, \qquad \tau > 0 \tag{A.2}$$

(Bingham *et al.*[10] give a good introduction to the theory of slowly varying functions.) From (4.2) we find

$$\lambda(\sigma, \omega, n) = \frac{n\lambda(\sigma\omega, \omega, n) + \tanh(\sigma)}{n\lambda(\sigma\omega, \omega, n)\tanh(\sigma) + 1} \tag{A.3}$$

Also for $n^{-1} > \omega$ or $\omega \geqslant n$ all excursions have finite duration and therefore

$$\lim_{\sigma \to 0} k(\sigma, 1, \omega, n) = 0$$

Hence

$$\lim_{\sigma \to 0} \sigma\lambda(\sigma, \omega, n) = 0 \tag{A.4}$$

The Case $\omega \geqslant n$. Since $\omega > 1$, the tree can be considered as an infinite line with drift away from O. Hence for fixed σ the local-time rate of marked excursions into the tree is greater than or equal to the local-time rate of marked excursions into an infinite line without drift (as the drift tends to increase the duration of an excursion), i.e.,

$$k(\sigma, 1, \omega, n) \geqslant \sigma$$

giving

$$\lambda(\sigma, \omega, n) \geqslant 1 \tag{A.5}$$

From (A.3)

$$\lim_{\sigma \to 0} \frac{\lambda(\sigma, \omega, n)}{\lambda(\sigma\omega, \omega, n)} = \lim_{\sigma \to 0} \left(\frac{n + \tanh(\sigma)/\lambda(\sigma\omega, \omega, n)}{n\lambda(\sigma\omega, \omega, n)\tanh(\sigma) + 1} \right)$$

and using (A.4) and (A.5) gives

$$\lim_{\sigma \to 0} \frac{\lambda(\sigma, \omega, n)}{\lambda(\sigma\omega, \omega, n)} = n \tag{A.6}$$

Substituting (A.1) into (A.6) gives

$$\lim_{\sigma \to 0} \frac{f(\sigma^{-1}, \omega, n)}{f(\sigma^{-1}\omega^{-1}, \omega, n)} = n\omega^{-\alpha}$$

and therefore (A.2) implies

$$\alpha = \frac{\log n}{\log \omega} \leqslant 1 \qquad \text{for} \quad \omega \geqslant n$$

Substituting (A.1) into (3.10) and expanding about $\sigma = 0$ gives

$$\begin{aligned} \mathbb{E}^{o}(W_M^2) &\sim \frac{4a\sigma^{\alpha-1}}{f(\sigma^{-1}, \omega, n)} \qquad \text{as} \quad \sigma \to 0 \\ &= a\Gamma\left(\frac{3-\alpha}{2}\right) \mu^{(\alpha-1)/2} g(\mu^{-1}, \omega, n) \qquad \text{as} \quad \mu \to 0 \end{aligned} \tag{A.7}$$

where

$$g(t, \omega, n) \equiv \frac{2^{(\alpha+3)/2}}{\Gamma((3-\alpha)/2)\, f((t/2)^{1/2}, \omega, n)}$$

It is easy to show that, for fixed ω and n, $g(\cdot, \omega, n)$ is slowly varying and since $\alpha \leqslant 1$ we can apply Karamata's Tauberian theorem (Theorem 1.7.1 in ref. 10) to (A.7), giving

$$\mathbb{E}^{o}(W_t^2) \sim a t^{(1-\alpha)/2} g(t, \omega, n) \qquad \text{as} \quad t \to \infty \tag{A.8}$$

The Case $\omega < n^{-1}$. From (A.3) and (A.4) we obtain

$$\lim_{\sigma \to 0} \frac{\lambda(\sigma, \omega, n)}{\lambda(\sigma\omega, \omega, n)} = n + \lim_{\sigma \to 0} \frac{\tanh(\sigma)}{\lambda(\sigma\omega, \omega, n)} \tag{A.9}$$

Substituting (A.1) and (A.2) into (A.9) gives

$$\lim_{\sigma \to 0} \frac{\sigma^{\alpha} \tanh(\sigma)}{f(\sigma^{-1}\omega^{-1}, \omega, n)} = 1 - n\omega^{-\alpha} \tag{A.10}$$

If $\alpha < -1$, then, by Proposition 1.3.6 of ref. 10,

$$\lim_{\sigma \to 0} \frac{\sigma^{\alpha} \tanh(\sigma)}{f(\sigma^{-1}\omega^{-1}, \omega, n)} = \infty$$

from which we conclude that $\alpha \geqslant -1$. If $\alpha > -1$, then, by Proposition 1.3.6 of ref. 10,

$$1 - n\omega^{-\alpha} = 0$$

giving

$$\alpha = -\log n / \log \omega^{-1}$$

Since k is define solely in terms of the radial part of the diffusion, we may evaluate it at noninteger values of n. Thus, for all n_1 and n_2 such that $\omega^{-1} > n_1 > n_2$ we find

$$\lim_{\sigma \to 0} \frac{k(\sigma, \omega, 1, n_1)}{k(\sigma, \omega, 1, n_2)} = 0 \tag{A.11}$$

using Proposition 1.3.6 of ref. 10. However, the drift away from 0 increases as n increases and therefore for all $\sigma > 0$

$$\frac{k(\sigma, \omega, 1, n_1)}{k(\sigma, \omega, 1, n_2)} \geqslant 1$$

which contradicts (A.11); hence $\alpha = -1$ and

$$\lim_{\sigma \to 0} (f(\sigma^{-1}\omega^{-1}, \omega, n))^{-1} = 1 - n\omega$$

Therefore

$$k(\sigma, 1, \omega, n) \sim \frac{\sigma^2}{1 - n\omega} \qquad \text{as} \quad \sigma \to 0 \tag{A.12}$$

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